The Structural Method

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Outline

1. Physical and Structural equations
2. Structural method for 1D (space discretization)
3. Structural method for ODE (time discretization)
4. One dimensional non-stationary equations

From compact schemes to structural schemes

Physical equations vs Structural equations

Consider the simple one-dimensional C-D-R equations

$$-\kappa\phi'' + u\phi' + r\phi = f, \quad \text{on } [0,1]$$

with Robin condition

$$\alpha(x)\phi'(x) + \beta(x)\phi(x) = g(x), \quad x = 0, 1.$$

Grid $x_i = i\Delta x$, $i = 0, \cdots, I$. A popular centered FD scheme reads

$$-\kappa\frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta x^2} + u\frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x} + r\phi_i = f_i$$

with $\phi_i \approx \phi(x_i)$.

Physical equations vs Structural equations

Rewrite with $Z_i \approx \phi(x_i)$, $D_i \approx \phi'(x_i)$, $S_i \approx \phi''(x_i)$.

$$-\kappa S_i + u D_i + r Z_i = f_i, \quad \text{(1st Physical Equation)}$$

with Robin condition (2nd Physical Equation)

$$\begin{aligned} \alpha_N(0) D_0 + \alpha_D(0) Z_0 &= g(0), \\ \alpha_N(1) D_I + \alpha_D(1) Z_I &= g(1), \end{aligned}$$

and

$$\begin{aligned} S_i &= \frac{Z_{i+1} - 2Z_i + Z_{i-1}}{\Delta x^2}, \quad \text{(1st Structural Equation)} \\ D_i &= \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x}, \quad \text{(2nd Structural Equation)}. \end{aligned}$$

☞ PE only deals with physics, SE only deals with mesh structure.

Implicit and explicit Structural equations

- ☞ Change of scheme ⇔ change the SEs (PEs are the sames).
- ☞ Use implicit SE ⇒ some classical compact schemes (HOC).

Hermitian compact scheme (Orzar & Israeli 1974, Adam 1975,)

$$\phi(x_{i+1}) - \phi(x_{i-1}) = \int_{x_{i-1}}^{x_{i+1}} \phi'(s)ds \Rightarrow$$

$$Z_{i+1} - Z_{i-1} = \frac{\Delta x}{6}(D_{i-1} + 4D_i + D_{i+1}). \qquad (SE1)$$

$$\frac{\phi(x_{i+1}) + \phi(x_{i-1}) - 2\phi(x_i)}{\Delta x^2} = \phi''(x_i) + \frac{\Delta x^2}{12}\phi^{(4)}(x_i) + O(\Delta x^4)$$

$$\frac{\phi''(x_{i+1}) + \phi''(x_{i-1}) - 2\phi''(x_i)}{\Delta x^2} = \phi^{(4)}(x_i) + O(\Delta x^2) \Rightarrow$$

$$\frac{Z_{i+1} + Z_{i-1} - 2Z_i}{\Delta x^2} = \frac{S_{i-1} + 10S_i + S_{i+1}}{12}. \qquad (SE2)$$

More on Hermitian scheme

☞ Philosophy (70s-90s): we see D or S as functions of Z to produce an implicit scheme with respect to Z.
☞ A lot of variants to increase the order and improve the spectral resolution (Kreiss & al. 1986, Lele 1992).
Typically the Hermitian schemes read

$$\sum_{k=-2}^{2} a_k Z_{i+k} = \sum_{k=-2}^{2} b_k D_{i+k}, \quad (SE1)$$

$$\sum_{k=-2}^{2} c_k Z_{i+k} = \sum_{k=-2}^{2} d_k S_{i+k}. \quad (SE2)$$

Together with (PE1), we have 3 Equations and 3 Unknowns by node (let's forget the BCs for the moment).

We have a configuration (PE1)-(SE1)-(SE2)

The Combined Compact Difference scheme

☞ Chu and Fan 1997, 1998 introduce two major ideas:

1. don't see D and S as function of Z but as real Unknowns of the problem (with the same status as Z);
2. a Structural Equation is a combination of Z, D and S over a small stencil.

SE equations take the form

$$\sum_{k=-1}^{1} a_k Z_{i+k} + \sum_{k=-1}^{1} b_k D_{i+k} + \sum_{k=-1}^{1} c_k S_{i+k} = 0.$$

Coefficients are determined using Taylor expansions and we eliminate the maximum number of terms.

☞ C& F produce 2 structural equations leading to a 6th order compact scheme.

☞ Excellent spectral resolution. No dissipation, very few dispersion.

☞ Since the 2000s', a lot of variations (non-uniform, larger stencil, system,...)

Structural method philosophy

1. Each node (or cell) contains several information: Z_i, D_i, S_i.
2. Physical Equations are local (no coupling between nodes).
3. Structural Equations linearly connect information of the neighbour nodes independently of the physics.
4. We need the same number of equations than the number of unknowns: mixe PEs and SEs.
5. Build families of SE (Hermitian and CCD are particular cases).
6. Not necessarily on the base of accuracy (stability, positivity, spectral resolution).
7. The choice of PEs and SEs may be different form one node to another one (BCs treatment).

The structural method for 1D problem

Framework

- $\Omega =]0,1[$ with grid $x_0 = 0$, x_i, $x_I = 1$.
- We introduce a functional to design three-points 9-coefs SEs.

$$\begin{aligned}\mathcal{E}_i(\boldsymbol{a};\phi) &= \sum_{k=-1}^{k=1} a_{i,k}^{(0)} \phi^{(0)}(x_{i+k}) \\ &+ \sum_{k=-1}^{k=1} a_{i,k}^{(1)} \phi^{(1)}(x_{i+k}) \\ &+ \sum_{k=-1}^{k=1} a_{i,k}^{(2)} \phi^{(2)}(x_{i+k})\end{aligned}$$

☞ Prescribe conditions on $\mathcal{E}_i$ to provide different sets of coefficients $\boldsymbol{a}$ (i.e. different SEs).

Structural Equations only based on accuracy

step 1 Write the 9x9 system such that $\mathcal{E}_i(\boldsymbol{a};\phi)=0$ for $\phi=(x-x_i)^m$, $m=0,\cdots,8$.

step 2 We derive the matrix system $M\boldsymbol{a}=0$ and Ker$(M)=0$.

step 3 Ker$(M[1:8,:])$ is a 1-dimension subspace with non-null vector $\boldsymbol{a_i^1}$. SEi1 equation is given by the non-degenerated equation

$$0 \;=\; \sum_{k=-1}^{k=1} a_{i,k}^{1,(0)} Z_{i+k} + \sum_{k=-1}^{k=1} a_{i,k}^{1,(1)} D_{i+k} + \sum_{k=-1}^{k=1} a_{i,k}^{1,(2)} S_{i+k}.$$

☞ For uniform mesh, we find again the first CCD of Chu-Fan equation and $\boldsymbol{a_i^1}$ does not depend on i.

Structural Equations only based on accuracy

step 4 Ker$(M[1:7,:])$ is a 2-dimension subspace that contains $\boldsymbol{a_i^1}$.
We take $\boldsymbol{a_i^2}$ orthogonal to $\boldsymbol{a_i^1}$ and obtain SEi2.

$$0 = \sum_{k=-1}^{k=1} a_{i,k}^{2,(0)} Z_{i+k} + \sum_{k=-1}^{k=1} a_{i,k}^{2,(1)} D_{i+k} + \sum_{k=-1}^{k=1} a_{i,k}^{2,(2)} S_{i+k}.$$

☞ For uniform mesh, we find again the second CCD of Chu-Fan equation and $\boldsymbol{a_i^2} = \boldsymbol{a^2}$.

☞ We loose one-order!

step 5 SEi3 derives from Ker$(M[1:6,:])$ where $\boldsymbol{a_i^3}$ is orthogonal to $\boldsymbol{a_i^1}$ and $\boldsymbol{a_i^2}$

☞ Does not exist in the C-F approach.

☞ We loose one more order!

Solving 1D C-D-R problem

Assume we have a uniform mesh with constant Δx.

- PE1(i): $-\kappa S_i + uD_i + rZ_i = f_i$, $i = 0, \cdots, I$. Including $i = 0$ and $i = I$
- PE2(L): $\alpha_L D_0 + \beta_L Z_0 = g(0)$
- PE2(R): $\alpha_R D_I + \beta_R Z_I = g(1)$
- SE1(i):

$$15(Z_{i+1} - Z_{i-1}) - \Delta x\Big(7D_{i+1} + 16D_i + 7D_{i-1}\Big) + \Delta^2 x\Big(S_{i+1} - S_{i-1}\Big) = 0.$$

- SE2(i):

$$24\,(Z_{i+1} - 2Z_i + Z_{j-1}) - 9\Delta x\Big(D_{i+1} - D_{i-1}\Big) + \Delta^2 x\Big(S_{i+1} - 8S_i + S_{i-1}\Big) = 0.$$

- SE3(i):

$$8\,(Z_{i+1} - 2Z_i + Z_{j-1}) - 5\Delta x\Big(D_{i+1} - D_{i-1}\Big) + \Delta^2 x\Big(S_{i+1} + S_{i-1}\Big) = 0.$$

And now, we cook the scheme!

Solving 1D C-D-R equation

- $i = 0$ we take PE1(0)+PE2(L)+SE3(1)
- $i = 1, \cdots, I$, we take PE1(i)+SE1(i)+SE2(i)
- $i = I$, we take PE1(I)+PE2(R)+SE3(I-1)

Benchmark. $\kappa = 1$, $u = 1$, $r = 1$, exact solution $\phi(x) = \exp(2x)$

☞ Dirichlet condition on both side ($\alpha = 1$, $\beta = 0$)

I	err(Z)	order	err(D)	order	err(S)	order	cond
10	3,18E-09	***	1,92E-08	***	9,33E-09	***	2,93E+02
20	5,24E-11	5,9	3,37E-10	5,8	1,66E-10	5,8	5,96E+02
40	8,51E-13	5,9	5,58E-12	5,9	2,77E-12	5,9	1,60E+03
80	2,85E-14	4,9	2,38E-13	4,6	1,91E-12	5,4	6,40E+03

☞ Same order with other kind of boundary ($\alpha = 0$, $\beta = 1$)

Solving 1D Bürger $\phi\ \phi' - \kappa\phi'' = f$

- PE1(i): $Z_i D_i - \kappa S_i = f_i$, $i = 0, \cdots, I$.
- PE2(L): $\alpha_L D_0 + \beta_L Z_0 = g(0)$, PE2(R): $\alpha_R D_I + \beta_R Z_I = g(1)$
- Same SE1, SE2, SE3

Fix point with a sequence of C-V $Z_i^{[\ell]} D_i^{[\ell+1]} - \kappa S_i^{[\ell+1]} = f_i$

$\kappa = 1$

I	err(Z)	order	err(D)	order	err(S)	order
10	1,13E-08	***	1,77E-07	***	5,44E-08	***
20	2,06E-10	5,9	3,35E-09	5,7	1,31E-09	5,3
40	3,50E-12	5,9	5,87E-11	5,8	2,63E-11	5,6
60	7,19E-13	4.9	4,04E-12	6,6	6,19E-12	3,5
80	5,37E-12	***	4,54E-11	***	4,44E-11	***

$\kappa = 1/1000$

I	err(Z)	order	err(D)	order	err(S)	order
10	8,50E-11	***	1,17E-07	***	5,64E-07	***
20	1,40E-12	5,9	2,13E-09	5,7	1,38E-08	5,3
40	3,20E-14	5,4	3,54E-11	5,9	3,86E-10	5,1
60	1,29E-14	2,2	3,69E-12	5,5	9,98E-11	3,3
80	2,75E-14	***	2,34E-12	***	3,30E-10	***

Some conclusions for the 1D space problem

- Very versatile schemes: choose your variables, your stencil and cook.
- Z, D and S have the same order, we do not loose accuracy for the 1st and 2nd derivatives.
- we do not have to compute the coefficients by hand but use the kernels to deduce the SEs.
- Trade-off between PEs and SEs: several options at the same point.
- A lot to say about diffusion/dispersion properties.
- 2D and 3D at least for structured grids.

Structural method in time for EDO

Framework

We want to solve the scalar EDO equation

$$\phi'(t) = f(\phi(t), t), \qquad \phi(0) = \phi_0.$$

We denote by $t_n = n\Delta t$, $n = 0, \cdots$ and the approximations

- $Z_n \approx \phi(t_n)$, $D_n \approx \phi'(t_n)$, $S_n \approx \phi''(t_n)$
- Physical Equation PE1(n) $D_n = f(Z_n, t_n)$
- Physical Equation PE2(n) $S_n = \partial_z f(Z_n, t_n) D_n + \partial_t f(Z_n, t_n)$

☞ Build the Structural Equations.

The one-point Z-D scheme ZD1

step 1 We introduce the functional

$$\mathcal{E}_n(\boldsymbol{a};\phi) = a_0^{(0)}\phi(t_n) + a_1^{(0)}\phi(t_{n+1}) + a_0^{(1)}\phi'(t_n) + a_1^{(1)}\phi'(t_{n+1})$$

step 2 Build the 4×4 matrix M such that we have $\mathcal{E}_n(\boldsymbol{a};\phi) = 0$ for $\phi = (t - t_n)^m$, $m = 0, 1, 2, 3$

step 3 Take the vector $\boldsymbol{a} \in \ker(M[1:3,:])$ that gives the SE1

$$\frac{D_{n+1} + D_n}{2} - \frac{Z_{n+1} - Z_n}{\Delta t} = 0$$

☞ Coupling PE1 and SE1 provides the 2nd Crank-Nicholson Scheme.

The two-points Z-D scheme ZD2

We use an intermediate point at $t_{n+1/2}$.

step 1 Two Physical Equations PE1, PE2:

$$D_{n+1/2} = f(Z_{n+1/2}, t_{n+1/2}), \quad D_{n+1} = f(Z_{n+1}, t_{n+1})$$

step 2 The (6 coefficients) functional is

$$\mathcal{E}_n(\boldsymbol{a}; \phi) = \sum_{k=0,1/2,1} \left[a_k^{(0)} \phi(t_{n+k}) + a_k^{(1)} \phi'(t_{n+k}) \right]$$

step 3 Build M such that $\mathcal{E}_n(\boldsymbol{a}; \phi) = 0$, $\phi = (t - t_n)^m$, $m = 0, \cdots, 5$.

step 4 Take vector $\boldsymbol{a}^{\mathbf{1}} \in \ker(M[1:5, :])$.

step 5 Take vector $\boldsymbol{a}^{\mathbf{2}} \in \ker(M[1:4, :])$, orthogonal to $\boldsymbol{a}^{\mathbf{1}}$.

The two-points Z-D scheme ZD2

We obtain two Structural Equations

$$
\begin{aligned}
0 &= -\frac{Z_{n+1} - Z_n}{\Delta t} + \frac{D_n + 4D_{n+1/2} + D_{n+1}}{6}, \ SE1 \\
0 &= -\frac{Z_n - 2Z_{n+1/2} + Z_{n+1}}{\Delta t} + \frac{D_{n+1} - D_n}{4}, \ SE2
\end{aligned}
$$

Given Z_n, D_n, we compute $Z_{n+1/2}$, $D_{n+1/2}$, Z_{n+1}, D_{n+1}, using SE1, SE2, PE1, PE2

Main properties

The scheme is A-stable. No restriction on Δt

The scheme is 4th-order both for Z and D.

☞ We find again the one-step implicit-block method: Shampine and Watts (1969,1972) and Axelsson (1969).

The one-point Z-D-S scheme ZDS1

The unknowns are Z_{n+1}, D_{n+1} and S_{n+1}

step 1 Two Physical Equations: $D_{n+1} = f(Z_{n+1}, t_{n+1})$ PE1, $S_{n+1} = \partial_z f(Z_{n+1}, t_{n+1}) D_{n+1} + \partial_t f(Z_{n+1}, t_{n+1})$ PE2.

step 2 The (6 coefficients) functional is

$$\mathcal{E}_n(\boldsymbol{a}; \phi) = \sum_{k=0,1} \left[a_k^{(0)} \phi(t_{n+k}) + a_k^{(1)} \phi'(t_{n+k}) + a_k^{(2)} \phi''(t_{n+k}) \right]$$

step 3 Build M such that $\mathcal{E}_n(\boldsymbol{a}; \phi) = 0$, $\phi = (t - t_n)^m$, $m = 0, \cdots, 5$.

step 4 Take vector $\boldsymbol{a}^{\mathbf{1}} \in \ker(M[1:5, :])$.

The one-point Z-D-S scheme ZDS1

We obtain one Structural Equation SE1

$$0 \;=\; 12(Z_n - Z_{n+1}) + 6\lambda\Delta t(Z_n + Z_{n+1}) + (\lambda\Delta t)^2(Z_n - Z_{n+1}).$$

Given Z_n, D_n, we compute Z_{n+1}, D_{n+1}, S_{n+1} using SE1, PE1, PE2

Main properties

The scheme is A-stable. No restriction on Δt
The scheme is 4th-order for Z, D, and S. Less equations to solve, but the price is the new Physical Equation (hence more non-linear)

☞ We find again the multistep multi-derivative method: Lambert and Mitchell (1962, 1963).

The two-points Z-D-S scheme ZDS2

Unknowns are $Z_{n+1/2}$, $D_{n+1/2}$, $S_{n+1/2}$ and Z_{n+1}, D_{n+1}, S_{n+1}

step 1 4 Physical Equations.

step 2 A (9 coefficients) functional

step 3 Build M such that $\mathcal{E}_n(\boldsymbol{a};\phi)=0$, $\phi=(t-t_n)^m$, $m=0,\cdots,8$.

step 4 Take vector $\boldsymbol{a}^{\mathbf{1}} \in \mathrm{ker}(M[1:8,:])$.

step 5 Take vector $\boldsymbol{a}^{\mathbf{2}} \in \mathrm{ker}(M[1:7,:])$, orthogonal to $\boldsymbol{a}^{\mathbf{1}}$.

The two-points Z-D-S scheme ZDS2

We obtain two Structural Equation SE1, SE2

$$
\begin{aligned}
0 &= 16\frac{Z_{n+1} - 2Z_{n+1/2} + Z_n}{(\Delta t)^2} - 3\frac{D_{n+1} - D_n}{\Delta t} + \frac{S_{n+1} - 8S_{n+1/2} + S_n}{6} \\
0 &= 30\frac{Z_{n+1} - Z_n}{(\Delta t)^2} - \frac{7D_{n+1} + 16D_{n+1/2} + 7D_n}{\Delta t} + \frac{S_{n+1} - S_n}{2}.
\end{aligned}
$$

Given Z_n, D_n, S_n we compute the 6 unknowns using SE1, SE2, PE1, PE2, PE3, PE4

Main properties

The scheme is linearly unconditional stable. No restriction on Δt
The scheme is 6th-order for Z, D, and S.

☞ We find again the multi-derivative implicit-block method (Enright 1974, Sahi, Jator, Khan 2012 Ehigie 2014).

A genuine two-points Z-D-S scheme NZDS

☞ All the traditional schemes systematicaly substitute D and S by the PEs: 2 SEs and 4 PEs.

We design a scheme with 4 SEs and 2 PEs (no equivalent in EDO)

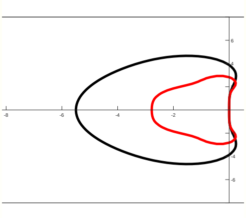
$$
\begin{aligned}
0 &= 16\frac{Z_{n+1} - 2Z_{n+1/2} + Z_n}{(\Delta t)^2} - 3\frac{D_{n+1} - D_n}{\Delta t} + \frac{S_{n+1} - 8S_{n+1/2} + S_n}{6},\\
0 &= 30\frac{Z_{n+1} - Z_n}{(\Delta t)^2} - \frac{7D_{n+1} + 16D_{n+1/2} + 7D_n}{\Delta t} + \frac{S_{n+1} - S_n}{2},\\
0 &= -8\frac{Z_{n+1} - 2Z_{n+1/2} + Z_n}{(\Delta t)^2} + \frac{D_{n+1} - D_n}{\Delta t} + S_{n+1/2},\\
0 &= 12\frac{Z_{n+1} - Z_n}{(\Delta t)^2} - 2\frac{D_{n+1} + 4D_{n+1/2} + D_n}{\Delta t},\\
D_{n+1} &= f(Z_{n+1}), \quad D_{n+1/2} = f(Z_{n+1/2}).
\end{aligned}
$$

☞ No Physical equation for S.

Main properties

The scheme is conditionally stable but with a larger stability region w.r.t. to RK4

The scheme is 4th-order for Z, D, and S.



☞ Additional note: with the Kernel technique, time step can be non-uniform.

Linear case

$$\phi' + \lambda\phi = f(t), \quad \phi(0) = \phi_0, \lambda \in \mathbb{R}$$

time scheme	N	Z		D		S	
		errZ	ord	errD	ord	errS	ord
2ZD	2	3.24E-05	---	3.24E-05	---		
	4	2.00E-06	4.02	2.00E-06	4.02		
	6	3.95E-07	4.01	3.95E-07	4.01		
	8	1.25E-07	4.00	1.25E-07	4.00		
1ZDS	2	3.24E-05	---	3.24E-05	---	3.24E-05	---
	4	2.00E-06	4.02	2.00E-06	4.02	2.00E-06	4.02
	6	3.95E-07	4.01	3.95E-07	4.01	3.95E-07	4.01
	8	1.25E-07	4.00	1.25E-07	4.00	1.25E-07	4.00
2ZDS	2	9.64E-09	---	9.64E-09	---	9.64E-09	---
	4	1.49E-10	6.02	1.49E-10	6.02	1.49E-10	6.02
	6	1.31E-11	6.01	1.31E-11	6.01	1.31E-11	6.01
	8	2.32E-12	6.00	2.32E-12	6.00	2.32E-12	6.00
NZDS	2	1.08E-05	---	1.08E-05	---	1.08E-05	---
	4	5.82E-07	4.22	5.82E-07	4.22	5.82E-07	4.22
	6	1.09E-07	4.13	1.09E-07	4.13	1.09E-07	4.13
	8	3.37E-08	4.09	3.37E-08	4.09	3.37E-08	4.09

Periodic case

$$\phi' + \mathrm{i}2k\pi\phi = f(t), \quad \phi(0) = \phi_0, \quad k = 10$$

time scheme	N	Z		D		S	
		err	ord	err	ord	err	ord
2ZD	10	1.80E+00	---	5.67E+01	---		
	20	2.27E-01	2.99	7.13E+00	2.99		
	30	4.91E-02	3.78	1.54E+00	3.78		
	40	1.60E-02	3.90	5.02E-01	3.90		
1ZDS	10	1.80E+00	---	5.67E+01	---	1.78E+03	---
	20	2.27E-01	2.99	7.13E+00	2.99	2.24E+02	2.99
	30	4.91E-02	3.78	1.54E+00	3.78	4.84E+01	3.78
	40	1.60E-02	3.90	5.02E-01	3.90	1.58E+01	3.90
2ZDS	10	2.64E-02	---	8.29E-01	---	2.61E+01	---
	20	6.74E-04	5.29	2.12E-02	5.29	6.65E-01	5.29
	30	6.42E-05	5.80	2.02E-03	5.80	6.34E-02	5.80
	40	1.18E-05	5.90	3.69E-04	5.90	1.16E-02	5.90
NZDS	10	1.94E+00	---	6.11E+01	---	1.92E+03	---
	20	7.31E-02	4.73	2.30E+00	4.73	7.22E+01	4.73
	30	1.37E-02	4.13	4.30E-01	4.13	1.35E+01	4.13
	40	4.25E-03	4.06	1.34E-01	4.06	4.20E+00	4.06

Non linear (stiff) case

Sigmoid $\phi(t) = \dfrac{1}{1+e^{-\lambda t}}$ of $\phi' = \lambda\phi(\phi-1)$, $\phi_0 = \phi(-1)$.

time scheme	N	Z		D		S	
	$\lambda = 5$	err	ord	err	ord	err	ord
2ZD	20	4.61E-02	---	7.40E+00	---		
	30	1.46E-02	2.83	1.57E+00	3.83		
	40	5.06E-03	3.70	5.08E-01	3.91		
	60	1.04E-03	3.90	1.02E-01	3.95		
1ZDS	20	4.61E-02	---	7.40E+00	---	4.64E+01	---
	30	1.46E-02	2.83	1.57E+00	3.83	1.47E+01	2.83
	40	5.06E-03	3.70	5.08E-01	3.91	5.09E+00	3.7
	60	1.04E-03	3.90	1.02E-01	3.95	1.05E+00	3.90
2ZDS	20	2.23E-04	---	2.17E-02	---	2.24E-01	---
	30	2.13E-05	5.79	2.07E-03	5.79	2.14E-02	5.79
	40	3.89E-06	5.90	3.80E-04	5.90	3.92E-03	5.90
	60	3.49E-07	5.95	3.41E-05	5.95	3.51E-04	5.95
2ZDS	10	4.81E-05	5.02	1.11E-04	6.12	7.69E-02	3.03
	20	3.04E-06	3.99	7.02E-06	3.99	1.85E-02	2.05
	30	6.01E-07	4.00	1.45E-06	3.88	8.17E-03	2.02
	40	1.90E-07	4.00	4.67E-07	3.95	4.58E-03	2.01

The (time) structural method for PDE

Space discretization

Consider the PDE

$$\partial_t \phi - \kappa \partial_{xx} \phi + u \partial_x \phi = f$$

with periodic boundary conditions.

- $A_c \phi$ the 6th/8th order finite difference discretization of $\partial_x \phi$
- $A_d \phi$ the 6th/8th order finite difference discretization of $\partial_{xx} \phi$
- Semi-discrete problem

$$\partial_t Z(t) - \kappa A_d Z(t) + u A_d Z(t) = F(t)$$

with $Z(t) = [Z_1(t), \cdots, Z_I(t)]^T$.

- With the same notations for $D(t)$ and $S(t)$, we have two Physical Equations

$$D(t) = \kappa A_d Z(t) - u A_d Z(t) + F(t), \quad PE1$$

$$S(t) = \kappa A_d D(t) - u A_d D(t) + F'(t), \quad PE2$$

Time discretization with Structural Equations

We present the ZDS1 case as an example.
For the EDO case, the structural equation reads

$$0 = 12(Z_n - Z_{n+1}) + 6\lambda\Delta t(D_n + D_{n+1}) + (\lambda\Delta t)^2(S_n - S_{n+1}).$$

We use exactly the same equation but as a vector with $Z_n = [Z_{n,1}, \cdots, Z_{n,I}]^T$

We solve the $3I$ system composed of PE1, PE2 and SE1.
☞ All the three schemes are unconditionally stable.

Benchmark pure convection equation

$$\partial_t \phi - \partial_x \phi = f$$

	N	Z				D				S			
		E1	O1	Eoo	Ooo	E1	O1	Eoo	Ooo	E1	O1	Eoo	Ooo
2ZD	20	6.76E-05	---	1.06E-04	---	4.25E-04	---	6.68E-04	---				
	25	2.78E-05	3.99	4.37E-05	3.99	1.74E-04	3.99	2.74E-04	3.99				
	30	1.34E-05	3.99	2.11E-05	3.99	8.42E-05	3.99	1.33E-04	3.99				
	35	7.24E-06	3.99	1.14E-05	3.99	4.55E-05	3.99	7.16E-05	3.99				
1ZDS	20	1.96E-04	---	3.08E-04	---	1.23E-03	---	1.94E-03	---	7.73E-03	---	1.22E-02	---
	25	8.04E-05	3.99	1.27E-04	3.99	5.05E-04	3.99	7.95E-04	3.99	3.18E-03	3.99	5.00E-03	3.99
	30	3.88E-05	3.99	6.11E-05	3.99	2.44E-04	3.99	3.84E-04	3.99	1.53E-03	3.99	2.41E-03	3.99
	35	2.10E-05	3.99	3.30E-05	3.99	1.32E-04	3.99	2.08E-04	3.99	8.28E-04	3.99	1.30E-03	3.99
2ZDS	20	1.02E-07	---	1.60E-07	---	6.38E-07	---	1.00E-06	---	4.00E-06	---	6.30E-06	---
	25	2.65E-08	6.02	4.18E-08	6.02	1.66E-07	6.03	2.61E-07	6.03	1.04E-06	6.05	1.63E-06	6.06
	30	8.77E-09	6.08	1.38E-08	6.08	5.45E-08	6.11	8.56E-08	6.12	3.35E-07	6.19	5.27E-07	6.19
	35	3.36E-09	6.22	5.29E-09	6.22	2.06E-08	6.33	3.22E-08	6.33	1.26E-07	6.33	1.99E-07	6.33

$$\partial_t \phi - \partial_{xx}\phi - \partial_x \phi = f$$

	N	Z				D				S			
		E1	O1	Eoo	Ooo	E1	O1	Eoo	Ooo	E1	O1	Eoo	Ooo
RK4	40	NaN	---	NaN	---								
	3700	NaN	---	NaN	---								
	3800	6.24E-11	---	9.81E-11	---								
2ZD	40	1.25E-05	---	1.96E-05	---	4.92E-04	---	7.73E-04	---				
	3700	7.08E-11	2.67	1.11E-10	2.67	9.30E-10	2.91	1.46E-09	2.91				
	3800	7.06E-11	0.09	1.11E-10	0.08	9.29E-10	0.05	1.46E-09	0.05				
1ZDS	40	3.33E-06	---	5.22E-06	---	1.31E-04	---	2.06E-04	---	5.18E-03	---	8.14E-03	---
	3700	7.06E-11	2.38	1.11E-10	2.38	9.33E-10	2.62	1.47E-09	2.62	1.24E-08	2.86	2.34E-08	2.82
	3800	7.04E-11	0.09	1.11E-10	0.07	9.31E-10	0.07	1.47E-09	0.06	1.24E-08	↑	2.47E-08	↑
2ZDS	40	4.45E-10	---	6.99E-10	---	1.64E-08	---	2.57E-08	---	6.82E-07	---	1.07E-06	---
	3700	7.06E-11	0.41	1.11E-10	0.41	9.31E-10	0.63	1.47E-09	0.63	1.24E-08	0.89	2.41E-08	0.84
	3800	7.04E-11	0.09	1.11E-10	0.07	9.30E-10	0.07	1.46E-09	0.05	1.23E-08	0.28	2.44E-08	↑

$$\partial_t \phi(t,x) - \mathrm{i}\partial_{xx}\phi(t,x) = f$$

	N	Z				D				S			
		E1	O1	Eoo	Ooo	E1	O1	Eoo	Ooo	E1	O1	Eoo	Ooo
2ZD	30	3.74E-05	---	3.61E-04	---	2.47E-04	---	2.47E-03	---				
	40	1.26E-05	3.80	1.16E-04	3.93	8.48E-05	3.71	8.02E-04	3.92				
	50	5.28E-06	3.89	4.83E-05	3.95	3.60E-05	3.84	3.33E-04	3.94				
	60	2.58E-06	3.93	2.35E-05	3.96	1.77E-05	3.90	1.62E-04	3.95				
	70	1.41E-06	3.94	1.27E-05	3.96	9.66E-06	3.92	8.81E-05	3.95				
1ZDS	30	3.74E-05	---	3.61E-04	---	2.47E-04	---	2.47E-03	---	1.87E-03	---	1.97E-02	---
	40	1.26E-05	3.80	1.16E-04	3.93	8.48E-05	3.71	8.01E-04	3.92	6.62E-04	3.61	6.40E-03	3.91
	50	5.28E-06	3.89	4.83E-05	3.95	3.60E-05	3.84	3.33E-04	3.94	2.84E-04	3.78	2.67E-03	3.92
	60	2.58E-06	3.93	2.35E-05	3.96	1.77E-05	3.90	1.62E-04	3.95	1.41E-04	3.86	1.30E-03	3.94
	70	1.41E-06	3.94	1.27E-05	3.96	9.66E-06	3.92	8.81E-05	3.95	7.70E-05	3.90	7.08E-04	3.95
2ZDS	30	6.26E-08	---	5.89E-07	---	5.61E-07	---	5.29E-06	---	5.58E-06	---	5.27E-05	---
	40	8.80E-09	6.82	8.53E-08	6.72	8.48E-08	6.56	8.18E-07	6.49	9.00E-07	6.34	8.61E-06	6.30
	50	2.73E-09	5.25	2.53E-08	5.44	1.78E-08	7.01	1.41E-07	7.88	1.57E-07	7.82	1.22E-06	8.75
	60	3.84E-09	↑	3.68E-08	↑	2.41E-08	↑	2.36E-07	↑	1.73E-07	↑	1.74E-06	↑
	70	4.48E-09	↑	4.13E-08	↑	2.97E-08	↑	2.78E-07	↑	2.29E-07	↑	2.16E-06	↑

Benchmark wave system

$$\partial_t \phi + \partial_x \psi = 0 \quad \partial_t \psi + \partial_x \phi = F$$

	N	Z				D				S			
		E1	O1	Eoo	Ooo	E1	O1	Eoo	Ooo	E1	O1	Eoo	Ooo
2ZD	20	1.45E-01	---	2.27E-01	---	4.54E+00	---	7.13E+00	---				
	40	1.02E-02	3.83	1.60E-02	3.83	3.20E-01	3.83	5.02E-01	3.83				
	80	6.54E-04	3.96	1.03E-03	3.96	2.06E-02	3.96	3.23E-02	3.96				
	160	4.12E-05	3.99	6.47E-05	3.99	1.29E-03	3.99	2.03E-03	3.99				
1ZDS	20	1.45E-01	---	2.27E-01	---	4.54E+00	---	7.13E+00	---	1.43E+02	---	2.24E+02	---
	40	1.02E-02	3.83	1.60E-02	3.83	3.20E-01	3.83	5.02E-01	3.83	1.00E+01	3.83	1.58E+01	3.83
	80	6.54E-04	3.96	1.03E-03	3.96	2.06E-02	3.96	3.23E-02	3.96	6.46E-01	3.96	1.01E+00	3.96
	160	4.12E-05	3.99	6.47E-05	3.99	1.29E-03	3.99	2.03E-03	3.99	4.06E-02	3.99	6.39E-02	3.99
2ZDS	20	4.29E-04	---	6.74E-04	---	1.35E-02	---	2.12E-02	---	4.23E-01	---	6.65E-01	---
	40	7.48E-06	5.84	1.18E-05	5.84	2.35E-04	5.84	3.69E-04	5.84	7.39E-03	5.84	1.16E-02	5.84
	80	1.20E-07	5.96	1.89E-07	5.96	3.77E-06	5.96	5.93E-06	5.96	1.19E-04	5.96	1.86E-04	5.96
	160	1.89E-09	5.99	2.97E-09	5.99	5.94E-08	5.99	9.33E-08	5.99	1.87E-06	5.99	2.93E-06	5.99

Benchmark Burger equation

$$\partial_t u - u\partial_x u = f$$

	N	$\bar{k}$	Z				D				S			
			E1	O1	Eoo	Ooo	E1	O1	Eoo	Ooo	E1	O1	Eoo	Ooo
2ZD	30	11.00	1.87E-05	---	4.39E-05	---	1.76E-04	---	4.13E-04	---				
	40	10.00	5.87E-06	4.03	1.38E-05	4.02	5.51E-05	4.03	1.31E-04	3.99				
	50	10.00	2.40E-06	4.02	5.64E-06	4.02	2.25E-05	4.02	5.35E-05	4.00				
	60	9.00	1.15E-06	4.01	2.71E-06	4.01	1.08E-05	4.01	2.58E-05	4.00				
1ZDS	30	13.00	3.43E-05	---	7.03E-05	---	2.17E-04	---	4.60E-04	---	3.57E-03	---	1.24E-02	---
	40	12.00	1.08E-05	4.00	2.22E-05	4.00	6.87E-05	4.00	1.46E-04	4.00	1.13E-03	3.99	3.95E-03	3.99
	50	11.00	4.43E-06	4.00	9.10E-06	4.00	2.81E-05	4.00	5.97E-05	4.00	4.65E-04	3.99	1.62E-03	4.00
	60	11.00	2.14E-06	4.00	4.39E-06	4.00	1.36E-05	4.00	2.88E-05	4.00	2.25E-04	4.00	7.81E-04	3.99
2ZD	30	11.97	7.91E-09	---	1.63E-08	---	5.01E-08	---	1.07E-07	---	8.29E-07	---	2.89E-06	---
	40	11.00	1.40E-09	6.01	2.89E-09	6.01	8.91E-09	6.01	1.89E-08	6.01	1.47E-07	6.00	5.13E-07	6.00
	50	10.98	3.68E-10	6.00	7.56E-10	6.01	2.33E-09	6.00	4.96E-09	6.01	3.87E-08	6.00	1.35E-07	5.99
	60	10.00	1.23E-10	6.00	2.53E-10	6.00	7.81E-10	6.00	1.66E-09	6.01	1.29E-08	6.00	4.51E-08	6.00

Benchmark Euler system

$$f = \partial_t \rho + \partial_x m, \quad g = \partial_t m + \partial_x (2\varepsilon + p), \quad h = \partial_t E + \partial_x (u(E+p))$$

	N	$\bar{k}$	Z_ρ				D_ρ				S_ρ			
			E1	O1	Eoo	Ooo	E1	O1	Eoo	Ooo	E1	O1	Eoo	Ooo
2ZD	10	9.00	8.30E-06	---	3.26E-05	---	9.32E-05	---	4.44E-04	---				
	15	8.20	1.65E-06	3.99	6.49E-06	3.98	1.85E-05	3.98	9.03E-05	3.93				
	20	7.70	5.21E-07	4.00	2.06E-06	3.99	5.88E-06	3.99	2.88E-05	3.97				
	25	7.40	2.14E-07	4.00	8.43E-07	4.00	2.41E-06	4.00	1.18E-05	3.99				
1ZDS	10	9.50	9.77E-06	---	1.78E-05	---	3.59E-05	---	1.45E-04	---	3.77E-04	---	1.51E-03	---
	15	8.80	1.93E-06	4.00	3.52E-06	4.00	7.12E-06	3.99	2.92E-05	3.96	7.51E-05	3.98	3.07E-04	3.93
	20	8.40	6.11E-07	4.00	1.11E-06	4.00	2.26E-06	3.99	9.29E-06	3.98	2.38E-05	3.99	9.80E-05	3.97
	25	8.00	2.50E-07	4.00	4.56E-07	4.00	9.25E-07	4.00	3.81E-06	3.99	9.77E-06	3.99	4.03E-05	3.98
2ZDS	10	8.60	7.26E-10	---	1.32E-09	---	2.67E-09	---	1.10E-08	---	2.81E-08	---	1.17E-07	---
	15	8.07	6.39E-11	6.00	1.16E-10	5.99	2.34E-10	6.00	9.61E-10	6.00	2.47E-09	6.00	1.02E-08	6.00
	20	7.80	1.14E-11	5.99	2.09E-11	5.97	4.07E-11	6.08	1.64E-10	6.15	4.29E-10	6.08	1.80E-09	6.04
	25	7.56	3.03E-12	5.93	5.68E-12	5.83	1.03E-11	6.17	4.22E-11	6.08	1.57E-10	4.49	5.78E-10	5.09

Conclusions

- Theoretical aspect (stability non linear, convergence)
- Connection with EDO scheme (generalized multi linear methods)
- Structural equation in time AND space
- Use other constrains than the accuracy (spectral resolution)
- 2D and 3D in space
- Generalized finite difference method on clouds
- Shock and discontinuities